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Buoyancy effects in plane Couette flow heated uniformly from below with constant heat flux

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Abstract

The onset of buoyancy-driven convection in plane Couette flow heated uniformly from below is analyzed theoretically. For the thermal entrance region with $Ra_q \gg 816.7$, a new set of stability equations involving streamwise variations of disturbances are derived based on the linear theory, the scaling relations and the extended momentary instability concept. It is shown that the critical Rayleigh number that marks the onset of longitudinal vortex rolls increases with a decrease in Prandtl number. Based on the present stability criteria a new mixed convection heat transfer correlation is derived for the whole range of Rayleigh number. The present analysis predicts the available experimental data of water, quite well.

Keywords: Buoyancy-driven instability; Heat transfer correlation; Turbulent convection; Plane Couette flow; Constant flux heating; Longitudinal vortex rolls; Propagation theory

1. Introduction

It is well known that buoyancy-driven secondary flow can play an important role in many engineering problems, such as chemical vapor deposition (CVD) and cooling of electronic equipments [1–3]. For a large Péclet number the secondary flow in the form of longitudinal vortex rolls has been observed in many of mixed convection experiments [4–9]. For the case of the thermal entrance region of uniformly heated plane Couette flow, Hung and Davis [10] first attempted to predict the onset of secondary vortex rolls by employing local stability theory where the basic temperature is frozen at each local position in the main flow direction. The streamwise variations of disturbances were first considered by Shin and Choi [11], and Choi et al. [12] proposed propagation theory by improving their previous approach. In propagation theory the thermal boundary-layer thickness was employed as a new characteristic length scaling factor and the disturbed quantities were transformed similarly. Choi and Kim [13,14] extended Choi et al.'s [12] original concept to the whole range of the Prandtl number and showed that for regular longitudinal vortex rolls the system becomes more unstable as the Prandtl number increases. Recently, Kim et al. [8,15] applied propagation theory to the plane Poiseuille flow heated from below. Their prediction compared reasonably with the existing experimental data.

To determine the critical conditions of time-dependent or axial-dependent convective instability problem, several criteria such as absolute, marginal and momentary criterion were suggested. An absolute stability criterion suggest the flow is unstable when or where the growth rate of disturbance energy is zero. In a marginal stability criterion, the flow is considered unstable when or where the disturbance energy reached its initial amplitude. Theses two criteria seems to be suitable for the initial-value approach such as amplification theory [16,17], where arbitrary initial disturbances are assumed

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Nomenclature

a	dimensionless wave number
d	fluid layer thickness
k	thermal conductivity
Nu	Nusselt number, $q_{\mathrm{w}}d/(k\Delta T)$
P	Pressure
p	dimensionless pressure disturbance
Pe	Péclet number, $U_{\mathrm{av}}d/\alpha$
Pr	Prandtl number, ν/α
q_{w}	bottom wall heat flux
Ra_q	Rayleigh number, $g\beta q_{\mathrm{w}}d^4/(k\alpha\nu)$
Re	Reynolds number, $U_{\mathrm{av}}d/\nu$
T	temperature
(U, V, W)	velocities in Cartesian coordinates
(u, v, w)	dimensionless velocity disturbances in Cartesian coordinates
(X, Y, Z)	Cartesian coordinates
(x, y, z)	dimensionless Cartesian coordinates

Greek symbols

α	thermal diffusivity
Δ_{T}	thermal boundary-layer thickness
δ_{T}	dimensionless thermal boundary-layer thickness
ζ	dimensionless similarity variable, $z/x^{1/3}$
θ	dimensionless temperature disturbance, $g\beta d^3 T_1/(\alpha\nu)$
θ_0	dimensionless basic temperature, $k(T_0 - T_i)/(q_{\mathrm{w}}d)$
λ	wave length of vortex roll
σ	temporal growth rate
ν	kinematic viscosity
τ	dimensionless time

Subscripts

0	basic quantities
1	perturbation quantities
c	critical conditions

Superscript

$*$	transformed quantities

and their amplification are monitored. The last one implies that the flow becomes unstable when or where the growth rate of the perturbation quantity (r_1) exceeds that of the base flow (r_0) [18].

Another important problem in natural convection will be its heat transfer characteristics in the thermally fully developed state. The possibility of connecting stability criteria to the heat transport on turbulent thermal convection in rapidly heated horizontal fluid layers was investigated by Howard [19] and modified by Busse [20]. Later, Long [21], Cheung [22] and Arpaci [23] proposed a backbone equation to predict heat transport in horizontal fluid layers by extending Howard's concept. By incorporating their stability criteria into the boundary-layer instability model, Choi and his colleagues have derived new heat transfer correlations for horizontal fluid layers [24], fluid-saturated horizontal porous layers [25] and plane Poiseuille flow [8] heated from below. Their resulting heat transfer correlations are in good agreement with a great deal of available experimental data.

In the present study, the buoyancy effects on plane Couette flow heated uniformly from below are analyzed theoretically. The critical conditions of thermal instability on regular longitudinal vortex modes are sought for the thermal entrance region with $Ra_q \gg 816.7$ and fully developed mixed convection heat transport correlation is derived for $Ra_q \geqslant 816.7$. Therefore, the present study is the extension and complement of the previous studies [8,11–15] for the buoyancy effects under the laminar channel flow.

2. Stability analysis

2.1. Basic flow and temperature fields

The basic system considered here is the plane Couette flow with a free upper boundary. The fluid layer of depth "d" over a horizontal plate is heated from below with constant heat flux q_{w}. The upper boundary is kept smooth at constant temperature T_r. The schematic diagram of the base system is shown in Fig. 1. The flow and temperature fields under forced convection can be expressed by the following dimensionless forms:

$$\overline{U}_0 = 2z \tag{1}$$

$$\overline{U}_0 \frac{\partial \theta_0}{\partial x} = \frac{\partial^2 \theta_0}{\partial z^2} + \frac{1}{Pe^2}\frac{\partial^2 \theta_0}{\partial x^2} \tag{2}$$

with inlet and boundary conditions,

$$\theta_0 = 0 \quad \text{at } x = 0 \quad \text{and} \quad z = 1 \tag{3}$$

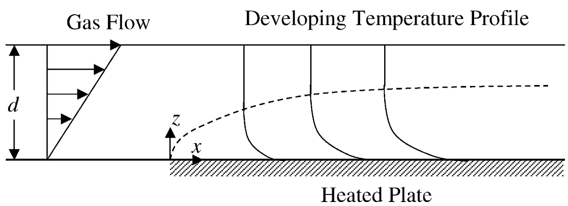

Fig. 1. Schematic diagram of the present system.

$$\frac{\partial \theta_0}{\partial z} = -1 \quad \text{at } z = 0 \tag{4}$$

where $(x,y,z) = (X/Pe, Y, Z)/d$, $\theta_0 = k(T - T_r)/(q_w d)$, and $\overline{U}_0 = U_0/U_{0,\mathrm{av}}$. k denotes the thermal conductivity, $U_{0,\mathrm{av}}$ is the average velocity and the subscript "0" denotes the base state. For large Péclet numbers, e.g., $Pe(= U_{\mathrm{av}} d/\alpha) > 100$, the last term in Eq. (2) is negligible. Then the Graetz-type solution, based on the method of separation of variables, is obtained as follows:

$$\theta_0 = 1 - z - \sum_{n=1}^{\infty} K_n R_n(z) S_n(z) \tag{5}$$

where

$$K_n = \frac{3^{4/3}}{\Gamma(2/3)\mu_n^{7/3} J_{2/3}^2(2\mu_n/3)}$$

$$R_n = z^{1/2} J_{-1/3}(2\mu_n z^{3/2}/3)$$

$$S_n = \exp(-\mu_n^2 x/2)$$

$$J_{-1/3}(2\mu_n/3) = 0$$

where $J_a(\omega)$ denotes the Bessel function of order a, and $\Gamma(\omega)$ the gamma function.

For $x \leqslant 0.05$ the following Leveque-type solution agrees well with the above Graetz-type solution [17]:

$$\theta_0 = \frac{(4.5x)^{1/3}}{\Gamma(2/3)} \left[\exp\left(-\frac{z^3}{4.5x} \right) - \frac{z}{(4.5x)^{1/3}} \Gamma\left(2/3, \frac{z^3}{4.5x} \right) \right]$$
$$= x^{1/3} \theta_0^*(\zeta) \quad \text{with } \delta_{\mathrm{T}} = 2.17 x_{\mathrm{c}}^{1/3} \tag{6}$$

where $\zeta = z/x^{1/3}$. Here δ_{T} denotes the dimensionless thermal boundary-layer thickness with $\theta_0^*(\zeta)/\theta_0^*(0) = 0.01$, $\Gamma(a,\omega)\left[= \int_\omega^\infty \exp(-t) t^{a-1}\,\mathrm{d}t\right]$ is an incomplete gamma function. The derivation of Eqs. (5) and (6) are described in the work of Choi [26].

2.2. Disturbance equations

By following the well-known linear stability analysis the infinitesimal perturbation quantities are superimposed on the basic quantities. The disturbances are usually assumed to be time-dependent, three-dimensional ones. For example, the dimensionless vertical velocity components w can be describe as

$$w = w^*(x,y,z) \exp[\mathrm{i}(a_x x + a_y y) + \sigma\tau] \tag{7}$$

where 'i' denotes the imaginary number, σ the temporal growth rate and τ the dimensionless time. With the longitudinal vortex roll the amplitude function w^* becomes independent of spanwise distance y with $a_x = 0$ and $\sigma = 0$ while the transverse roll brings $\sigma \neq 0$ with $a_y = 0$. For a horizontal channel flow heated from below Lin and his colleagues [2,3] conducted careful flow visualization and transient temperature measurements and showed that at a very low Reynolds number for $3.0 \leqslant Re \leqslant 5.0$, transverse rolls of $a_x \neq 0$ and $\sigma \neq 0$ and mixed transverse/longitudinal rolls were observed. Also, they suggested the flow regime as a function of the Reynolds number and the Rayleigh number. According to their results and many other experimental results [1–9,26], for large Péclet number case, near the critical position time-independent vortex rolls have been observed experimentally. Therefore, for large Péclet numbers the following disturbance equations can be obtained in dimensionless form by invoking linear theory under the Boussinesq approximation:

$$\frac{1}{Pr}\left\{ \overline{U}_0 \frac{\partial u}{\partial x} + w \frac{\partial \overline{U}_0}{\partial z} \right\} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{8}$$

$$\frac{1}{Pr}\left\{ \overline{U}_0 \left(\frac{\partial^3 u}{\partial x^2 \partial z} + \frac{\partial^3 w}{\partial x \partial z^2} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \right\}$$
$$= \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)^2 w + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^4 u}{\partial x \partial y^2 \partial z} + \frac{\partial^4 w}{\partial x \partial z^3} \tag{9}$$

$$\overline{U}_0 \frac{\partial \theta}{\partial x} + Ra_q \left\{ u \frac{\partial \theta_0}{\partial x} + w \frac{\partial \theta_0}{\partial z} \right\} = \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \tag{10}$$

with the boundary conditions,

$$u = w = \frac{\partial w}{\partial z} = \frac{\partial \theta}{\partial z} = 0 \quad \text{at } z = 0 \tag{11a}$$

$$u = w = \frac{\partial^2 w}{\partial z^2} = \theta = 0 \quad \text{at } z = 1 \tag{11b}$$

where $(u,w) = (U_1/Pe, W_1)d/\alpha$ and $\theta = g\beta d^3 T_1/(\alpha\nu)$. α denotes the thermal diffusivity, g the gravity acceleration, β the thermal expansivity, and ν the kinematic viscosity. It should be noted that the temperature disturbance is nondimensionalized by $\alpha\nu/(g\beta d^3)$ which is widely used disturbance temperature scale rather than ΔT [16]. Here, the Prandtl bumber Pr and Rayleigh number based on the bottom heat flux q_w, Ra_q are defined as

$$Pr = \frac{\nu}{\alpha} \quad \text{and} \quad Ra_q = \frac{g\beta q_w d^4}{k\alpha\nu} \tag{12}$$

To examine the thermal instability of the present system we must find the minimum value of Ra_q to satisfy Eqs. (8)–(11). Most of previous studies on this kind of stability problems have employed the assumption that disturbances would not experience variations in the streamwise direction, i.e. $\partial(\cdot)/\partial x = 0$. In the present study this assumption is removed by employing our propagation theory. This theory takes the streamwise propagation of disturbances into consideration.

2.3. Propagation theory

The propagation theory employed to find the dimensional critical streamwise position X_c to mark the onset of convective motion is based on the assumption that disturbances are propagated mainly within the dimensional thermal boundary-layer thickness $\Delta_T (\ll d)$ at $X_c \gg \Delta_T$. In this case the following scale analysis at $X \approx X_c$ would be valid for dimensionless perturbed quantities of Eqs. (9) and (10), respectively.

$$\frac{w}{\delta_T^2} \sim \theta \tag{13}$$

$$Ra_q w \frac{\partial \theta_0}{\partial z} \sim \frac{\theta}{\delta_T^2} \tag{14}$$

This means that buoyancy-driven convection occurs due to θ and this incipient secondary flow is very weak at $x = x_c$. Because $\partial\theta_0/\partial z|_{x=x_c}$ has the magnitude of order of 1, $Ra_q \delta_T^4$ is a constant for $\delta_T \ll 1$ from the above relations. In this viewpoint the base temperature and its perturbation have been nondimensionalized having different scales. The above scaling analysis describes in depth in Kim et al.'s [15]. Based on the above relations, the following relations can be obtained.

$$u = x\delta_T^{n+1} u^*, \quad w = \delta_T^{n+2} w^* \quad \text{and} \quad \theta = \delta_T^n \theta^* \tag{15}$$

At this stage the criterion to determine n is necessary. Shen [18] suggested the momentary instability condition: the temporal growth rate of the perturbation quantities (r_1) should exceed that of the base flow (r_0). By extending this conception into the present system, the critical condition can be determined at a certain position where $r_0 = r_1$. In the present system the dimensionless streamwise growth rates are defined as quantities of temperature components:

$$r_0 = \frac{1}{\langle\theta_0\rangle} \frac{\mathrm{d}\langle\theta_0\rangle}{\mathrm{d}x} \quad \text{and} \quad r_1 = \frac{1}{\langle\theta_1\rangle} \frac{\mathrm{d}\langle\theta_1\rangle}{\mathrm{d}x} \tag{16}$$

where $\langle\text{quantity}\rangle = \left[\left(\int_A (\text{quantity})^2 \mathrm{d}A\right)/A\right]^{1/2}$ and $\mathrm{d}A = \lambda \mathrm{d}z$. Here A denotes the cross sectional area of one vortex roll pair in x–y plane. From the base temperature distribution of Eq. (6), r_0 can be obtained as:

$$r_0 = \frac{1}{4x} \quad \text{for} \quad x \to 0. \tag{17}$$

For the case of $n = 1$, the condition of $r_0 = r_1$ is fullfiled at $x = x_c$. If the laminar-forced convection is still dominant with $Ra^* =$ constant at $x = x_c$, it is probable that $\theta(x,z) = x^{1/3}\theta^*(\zeta)$. This means that the amplitude function of temperature disturbances follows the behavior of θ_0 for small x.

Then for the longitudinal vortex rolls the disturbance quantities are expressed as

$$\begin{bmatrix} u(x,y,z) \\ w(x,y,z) \\ \theta(x,y,z) \end{bmatrix} = \begin{bmatrix} x^{5/3}u^*(\zeta) \\ xw^*(\zeta) \\ x^{1/3}\theta^*(\zeta) \end{bmatrix} \exp(\mathrm{i}ay) \tag{18}$$

where the superscript "*" refers to the amplitude function. It is noted that the dimensionless thermal boundary-layer thickness has the order of magnitude of $x^{1/3}$ and each quantity in Eqs. (5), (6) and (21) is based on this vertical length scale.

Substituting Eq. (18) into Eqs. (8)–(11), we can obtain the new stability equations:

$$(D^2 - a^{*2})u^* = \frac{1}{Pr}\left(\frac{10}{3}\zeta u^* - \frac{2}{3}\zeta^2 Du^* + 2w^*\right) \tag{19}$$

$$\begin{aligned}(D^2 - a^{*2})^2 w^* &= a^{*2}\theta^* - \frac{2}{3}D^3u^* + \frac{1}{3}\zeta D^4 u^* + \frac{4}{3}a^{*2}Du^* - \frac{1}{3}\zeta a^{*2}D^2u^* \\ &\quad + \frac{1}{Pr}\left\{\frac{4}{3}\left(\frac{5}{3}u^* - \frac{1}{3}\zeta Du^* + Dw^*\right)\right. \\ &\quad - \frac{2}{3}\zeta^2\left(Du^* - \frac{1}{3}\zeta D^3u^* + D^3w^*\right) \\ &\quad \left. - 2\zeta a^{*2}w^* + \frac{2}{3}\zeta^2 a^{*2}Dw^*\right\},\end{aligned} \tag{20}$$

$$\begin{aligned}(D^2 - a^{*2})\theta^* &= \frac{2}{3}\zeta\theta^* - \frac{2}{3}\zeta^2 D\theta^* + Ra^*\left(w^*D\theta_0^*\right. \\ &\quad \left. - \frac{1}{3}\zeta u^* D\theta_0^* + \frac{1}{3}u^*\theta_0^*\right)\end{aligned} \tag{21}$$

with boundary conditions,

$$u^* = w^* = Dw^* = D\theta^* = 0 \quad \text{at } \zeta = 0 \tag{22a}$$

$$u^* = w^* = D^2w^* = \theta^* = 0 \quad \text{as } \zeta \to \infty \tag{22b}$$

where $D = \frac{\mathrm{d}}{\mathrm{d}\zeta}$, $a^* = ax^{1/3}$, and $Ra^* = Ra_q x^{4/3}$. The parameters a^* and Ra^* based on the length scaling factor $x^{1/3}$ are assumed to be eigenvalues. Now, the principle of exchange of stabilities is employed and the minimum value of Ra^* for a given Pr is sought. This whole procedure is the essence of our propagation theory.

The above stability equations are quite different from those of Choi and Kim [14]. Without any deterministic criterion they set $n = 0$ in Eq. (15), that is the temperature disturbances was set to $\theta(x,z) = \theta^*(\zeta)$ and assumed to be follow the behavior of $\theta_0^*(\zeta)$. This is the major difference between the work of Choi and Kim [14] and the present one. In the present study, based on the momentary-instability concept the amplitude function of temperature disturbances are assumed to follow the behavior of θ_0 for small x.

2.4. Solution method

To find eigenvalues and eigenfunctions for differential equations, several methods such as compound matrix method and shooting method are proposed [27]. In the present study the stability Eqs. (19)–(22) are solved by employing the latter method. For a specific value of eigenvalue, many other eigenfunctions which are different by constant ratio are possible. To determine the eigenvalue, unprescribe initial value can be assigned arbitrarily. Thereby the boundary value problem is converted as an initial value problem. This is the characteristics of eigenvalue problem.

To integrate these stability equations the proper values of Du^*, D^2w^*, D^3w^* and θ^* at $\zeta = 0$ are assumed for a given Pr and a^*. Since the stability equations and the boundary conditions are all homogeneous, the value of D^2w^* at $\zeta = 0$ can be assigned arbitrarily and the value of the parameter θ^* is assumed. After all the values at $\zeta = 0$ are provided, this eigenvalue problem can be proceeded numerically.

Integration is performed from the heated surface $\zeta = 0$ to a fictitious outer boundary with the fourth order Runge–Kutta–Gill method. If the guessed values of Ra^*, $Du^*(0)$, $D^3w^*(0)$ and $\theta^*(0)$ are correct, u^*, w^*, D^2w^* and θ^* will vanish at the outer boundary. To improve the initial guesses the Newton–Raphson iteration is used. When convergence is achieved, the outer boundary is increased by predetermined value and the above procedure is repeated. Since the disturbances decay exponentially outside the thermal boundary layer, the incremental change in Ra^* also decays fast with an increase in outer boundary depth. This behavior enables us to extrapolate the eigenvalue Ra^* to the infinite depth by using Shank transform as

$$Ra^*_\infty = Ra^*_{\zeta_3} + \frac{r}{1-r}(Ra^*_{\zeta_1} - Ra^*_{\zeta_2}) \tag{23}$$

where Ra^*_∞ is the value of Ra^* extrapolated to the infinite depth and $Ra^*_{\zeta_i}$ is the calculated value of Ra^* when the outer boundary is kept at $\zeta = \zeta_i$. The decay ratio r is defined as

$$r = (Ra^*_{\zeta_3} - Ra^*_{\zeta_2})/(Ra^*_{\zeta_2} - Ra^*_{\zeta_1}). \tag{24}$$

The whole solution procedure is described in the works of Chen and Chen [28], Kim [29] and Chen et al. [30].

2.5. Results of stability analysis

The predicted values based on the above numerical scheme constitute the stability curve, as shown in Fig. 2. From this figure the stability criteria of the minimum Ra^* for the case of $Pr = 7$ are obtained. In Table 1 the present results are compared with those of Choi and Kim [14]. The difference between two becomes smaller with an increase in Pr. For other Pr the present stability criteria are listed in Table 2.

For small Pr there are large differences between the present stability criteria and Choi and Kim's [14], as shown in Fig. 3. It seems evident that Ra^*_c increases with a decrease in Pr and the Pr effect becomes pronounced for $Pr \leqslant 1$. This trend can be shown clearly in Fig. 3. As $Pr \to 0$ we can expect that the other mode of instability such as wave mode may prevail and the present analysis cannot be applied.

In Figs. 4 and 5, the water data of Choi [26] are compared with the present critical conditions (Table 1):

$$x_c = 20.44 Ra_q^{-3/4} \quad \text{and} \quad a_c = 0.358 Ra_q^{1/4} \quad \text{for } Pr = 7. \tag{25}$$

It is known that in the thermal entrance region of $x \leqslant 0.05$ the present predictions represent the experimental data very well. For the present system experimental data except those of water are not known. But the present trend that a lower Pr fluid becomes more stable with respect to regular longitudinal roll modes looks reasonable, and the experimental results of Maughan and Incropera [9] for the case of plane Poiseuille flow also show this trend. Even the present prediction for the onset position is quite similar to that of Choi and Kim [14], the present critical wave number is more reasonable.

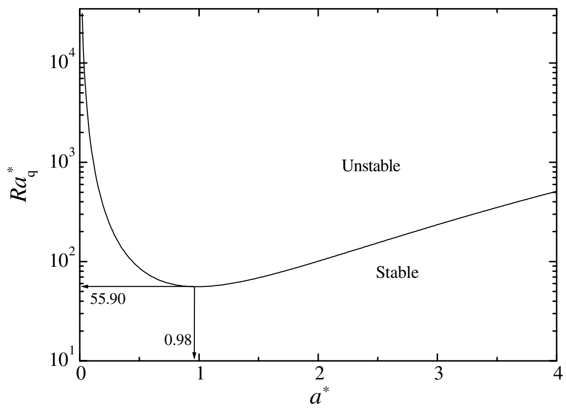

Fig. 2. Neutral stability curve for $Pr = 7$.

Table 1
Comparison of critical values for $Pr = 7$

Theoretical results	a^*_c	Ra^*_c
Present study	0.98	55.90
Choi and Kim [14]	0.87	55.49

Table 2
Numerical values of Ra^*_c and a^*_c for various Pr

Pr	0.01	0.1	0.7	1	10	100	∞
Ra^*_c	2907	300.7	104.7	90.91	227.93	47.11	46.23
a^*_c	1.61	1.33	1.18	1.15	1.23	0.88	0.87

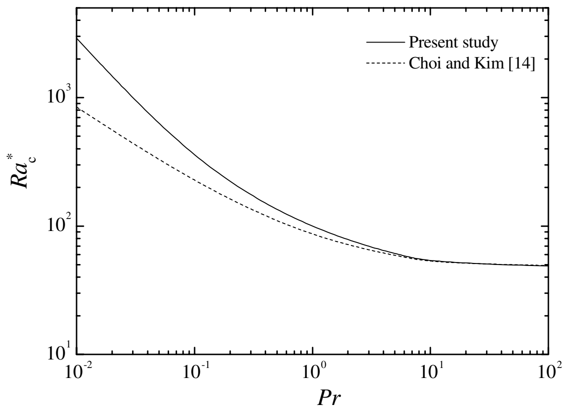

Fig. 3. Effect of Prandtl numbers on critical conditions.

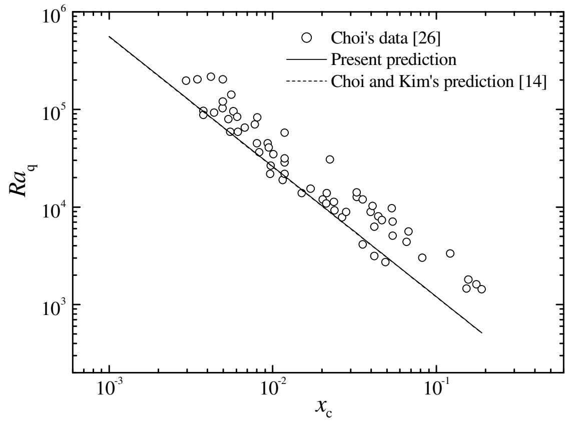

Fig. 4. Comparison of critical Rayleigh numbers with water data.

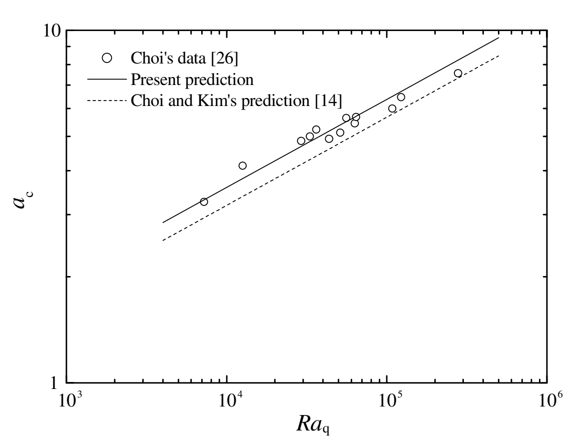

Fig. 5. Comparison of critical wave numbers data.

3. Heat transport

3.1. Turbulent heat transfer

Howard [19] assumed that turbulent heat transport would be governed by a narrow boundary layer like a conduction film of thickness δ_c near a heated surface, independently of the whole fluid-layer depth. δ_c is usually called the conduction layer thickness. Further, he suggested that the conduction period τ_* can be approximated as the onset time of transient convection τ_c, that is the conduction thickness δ_c may approximated as the thermal penetration depth at the onset of buoyancy-driven convection. Busse [20] modified Howard's concept such that heat transport resistances exist not only near the heated bottom surface but also near the cooling upper boundary. This so-called boundary-layer instability model is featured in Fig. 6. As shown in Fig. 6, the temperature difference over the conduction layer thickness is a half of the total temperature difference ΔT. By Busse [20] the Nusselt number in the fully developed turbulent state is given

$$Nu = \frac{Q_{\text{actual}}}{Q_{\text{conduction}}} = \frac{k\Delta T_\delta/\delta_c}{k\Delta T/d} = \frac{1}{2}\frac{d}{\delta_c} = \frac{1}{2}\left(\frac{Ra}{2Ra_\delta}\right)^{1/3} \tag{26}$$

where $Ra_\delta = g\beta\Delta T_\delta\delta^3/(\alpha\nu)$. Ra is the usual Rayleigh number based on ΔT, Ra_δ the Rayleigh number based on the conduction thickness, and ΔT_δ the temperature difference over the conduction layer thickness. For the uniformly heated system the above relationship can be modified by using the relationship of $Ra_q = RaNu$. Thus, in the present system Eq. (26) is replaced by

$$Nu = \frac{1}{2}\left(\frac{Ra_q}{Ra_\delta}\right)^{1/4} \quad \text{for } Ra_q \to \infty \tag{27}$$

Long [21] and Cheung [22] analyzed the above turbulent heat transport and suggested the flowing relation:

$$Nu = \frac{ARa^{1/3}}{[1 - B(RaNu)^{-1/12}]^{4/3}}. \tag{28}$$

From above relation, it can be showed that the heat transfer characteristics for $Ra_q \to \infty$ would be independent of the fluid-layer depth like Howard and Bus-

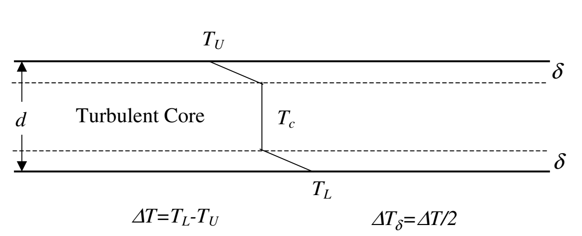

Fig. 6. Schematic diagram of the boundary-layer instability model.

se's concept. By using the relationship of $Ra_q = RaNu$ and slight modification of Eq. (28) produces the following heat transfer correlation for the present uniformly heated system:

$$Nu = 1 + \frac{A(Ra_q^{1/4} - 816.7^{1/4})}{1 - BRa_q^{-1/12}} \quad \text{for } Ra_q \geqslant 816.7 \tag{29}$$

where the value of 816.7 is the minimum Rayleigh number to mark buoyancy-driven convection. The constants A and B should be determined by using experimental data or theoretical relations. Therefore, Eq. (26) is the limiting case of Eqs. (28) and (29) for $Ra_q \to \infty$.

Now, by extending the Howard's concept to the present problem we assume that the conduction-layer thickness in the boundary-layer instability model could be approximated by the thermal boundary-layer thickness at the onset position of buoyancy-driven convection. Then from both the stability analysis results shown in Table 1 and the basic temperature field the following relations are obtained as a function of the critical position x_c for water with $Pr = 7$:

$$Nu = 0.0973Ra_q^{1/4} \quad \text{for } Ra_q \to \infty \tag{30}$$

3.2. Heat transfer correlation

The heat transport with Ra_q near $Ra_q = 816.7$ can be estimated by weakly nonlinear analysis base the shape assumption of Stuart [31] as

$$\frac{1}{Nu} = 1 - \frac{\Lambda}{Ra_{q,c}}(Ra_q - Ra_{q,c}) \quad \text{for } Ra_q \to 816.7 \tag{31}$$

where Λ is the constant calculated by using profiles of disturbances at $Ra_q = 816.7$. The value of Λ for the present system is obtained from the relation

$$\Lambda = \frac{\left(\int_0^1 w_1\theta_1\,dz\right)^2}{\int_0^1 (w_1\theta_1)^2\,dz} = 0.6722 \tag{32}$$

where w_1 and θ_1 are the dimensionless velocity and temperature disturbances at $Ra_q = 816.7$, respectively.

For the uniformly heated plane Couette flow of water a new heat transfer correlation for the whole range of the Rayleigh number satisfying Eqs. (29)–(32) is obtained with $Pr = 7$ as

$$Nu = 1 + \frac{0.0980(Ra_q^{1/4} - 816.7^{1/4})}{1 - 1.4345Ra_q^{-1/12}} \tag{33}$$

The above prediction suggests the lower bounds of the water data of Choi [26], as shown in Fig. 7. It is noted that $Nu \equiv 1$ for $Ra_q \leqslant 816.7$. This value corresponds to that of thermally fully developed forced convection. It is noted that by using the above procedure the corresponding correlation can be generated for each Pr case.

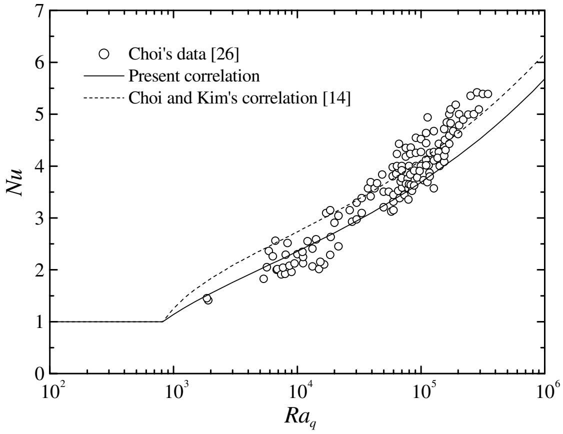

Fig. 7. Comparison of the present heat transfer correlation with water data.

4. Conclusion

The condition of the onset of regular longitudinal vortex rolls in the thermal entrance region of plane Couette flow heated uniformly from below has been analyzed theoretically. The theoretical analysis was conducted by using the propagation theory. Based on the propagation theory and the momentary-instability concept a new set of stability equations was derived and solved numerically. The resulting prediction showed that the critical Rayleigh number increases with a decrease in Prandtl number. Based on the present stability criteria, a new heat transfer correlation in mixed convection is suggested. It is shown that these theoretical results agree well with experimental data for water. Therefore it may be stated that our propagation theory may be useful for predicting the onset position of buoyancy-driven motion in laminar mixed convection flow and also for deriving the heat transport correlation in fully developed state as a function of the Rayleigh number and the Prandtl number. This study may be the complements of the results of Choi and Kim [14].

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